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ORIGINAL PAPER

Solving coupled Lane–Emden boundary value problems in catalytic diffusion reactions by the Adomian decomposition method

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Abstract In this paper, we consider the coupled Lane–Emden boundary value problems in catalytic diffusion reactions by the Adomian decomposition method. First, we utilize systems of Volterra integral forms of the Lane–Emden equations and derive the modified recursion scheme for the components of the decomposition series solutions. The numerical results display that the Adomian decomposition method gives reliable algorithm for analytic approximate solutions of these systems. The error analysis of the sequence of the analytic approximate solutions can be performed by using the error remainder functions and the maximal error remainder parameters, which demonstrate an approximate exponential rate of convergence.

Keywords Lane–Emden equation · Boundary value problem · Adomian decomposition method · Volterra integral equation

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1 Introduction

The Lane–Emden equation was first studied by astrophysicists Jonathan Homer Lane and Robert Emden, where they considered the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics [15]. The well–known Lane–Emden equation has been used to model several phenomena in mathematical physics and astrophysics such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas spheres, and the theory of thermionic currents. A substantial amount of work has been done on these types of problems for various structures [15,16,18–24]. The singular behavior that occurs at $r = 0$ is the main difficulty of the Lane–Emden equations.

Systems of Lane–Emden equations arise in the modelling of several physical phenomena, such as pattern formation, population evolution, chemical reactions, and so on [11,12,17,25].

In [24], the Adomian decomposition method (ADM) was used to solve the Volterra integral form of the Lane–Emden equation with initial values and boundary conditions. In [23], the initial value problem for the systems of the Volterra integral forms of the Lane–Emden equations was solved by the ADM.

The ADM [1–4,9,13,14,21] is a well–known systematic method for solving linear and nonlinear equations, including ordinary differential equations, partial differential equations, integral equations, integro-differential equations, etc. The method permits us to solve both nonlinear initial value problems and boundary value problems. For a comprehensive bibliography featuring many new engineering applications and a modern review of the ADM, see [9,14].

Duan and Rach [8] proposed a new modified recursion scheme for the resolution of multi-order and multi-point boundary value problems for nonlinear ordinary and partial differential equations by the ADM. The new approach, including Duan's convergence parameter [6,8,10], provides a significant computational advantage by allowing for the acceleration of convergence and expansion of the interval of convergence during calculations of the solution components for nonlinear boundary value problems. All of the boundary conditions were used before we derive a modified recursion scheme without any undetermined coefficients when computing successive solution components. This modification also avoids solving a sequence of nonlinear algebraic equations for the undetermined coefficients fraught with multiple roots, which is required to complete calculation of the solution by several prior modified recursion schemes using the ADM.

We aim in this work to apply Duan-Rach modified recursion scheme in the ADM to the mixed boundary value problems for system of the Lane–Emden equations. The coupled Lane–Emden boundary value problems in catalytic diffusion reactions are solved. We will show that using the integral form facilitates the computational work and overcomes the singularity behavior at $r = 0$. The error analysis can be performed by using the error remainder functions and the maximal error remainder parameters, which demonstrate an approximate exponential rate of convergence.

2 The Volterra integral form and the modified ADM

We consider the mixed boundary value problems for the coupled Lane–Emden equations

$$u''(r) + \frac{2}{r}u'(r) + f_1(u(r), v(r)) = 0, \tag{1}$$

$$v''(r) + \frac{2}{r}v'(r) + f_2(u(r), v(r)) = 0, \tag{2}$$

$$u'(0) = 0,\ u(1) = \beta_1,\ v'(0) = 0,\ v(1) = \beta_2. \tag{3}$$

Applying the equivalent Volterra integral forms for Eqs. (1) and (2), we obtain [23, 24]

$$u(r) = u(0) - \int_0^r \tau\left(1 - \frac{\tau}{r}\right) f_1(u(\tau), v(\tau))\, d\tau, \tag{4}$$

$$v(r) = v(0) - \int_0^r \tau\left(1 - \frac{\tau}{r}\right) f_2(u(\tau), v(\tau))\, d\tau. \tag{5}$$

Next we use algebraic manipulation to determine the values of the undetermined coefficients $u(0)$ and $v(0)$. Substituting the boundary values at $r = 1$ in Eq. (3) we have

$$\beta_1 = u(0) - \int_0^1 \tau(1 - \tau) f_1(u(\tau), v(\tau))\, d\tau,$$

$$\beta_2 = v(0) - \int_0^1 \tau(1 - \tau) f_2(u(\tau), v(\tau))\, d\tau.$$

Thus we obtain the unprescribed conditions

$$u(0) = \beta_1 + \int_0^1 \tau(1 - \tau) f_1(u(\tau), v(\tau))\, d\tau, \tag{6}$$

$$v(0) = \beta_2 + \int_0^1 \tau(1 - \tau) f_2(u(\tau), v(\tau))\, d\tau. \tag{7}$$

Upon substitution of $u(0)$ and $v(0)$ into Eqs. (4) and (5), we obtain the system of two-coupled nonlinear Fredholm-Volterra integral equations as

$$u(r) = \beta_1 + \int_0^1 \tau(1-\tau) f_1(u(\tau), v(\tau))\, d\tau - \int_0^r \tau\left(1-\frac{\tau}{r}\right) f_1(u(\tau), v(\tau))\, d\tau, \tag{8}$$

$$v(r) = \beta_2 + \int_0^1 \tau(1-\tau) f_2(u(\tau), v(\tau))\, d\tau - \int_0^r \tau\left(1-\frac{\tau}{r}\right) f_2(u(\tau), v(\tau))\, d\tau. \tag{9}$$

Applying the Adomian decomposition series and the series of the two-variable Adomian polynomials [4,5,7,13], we have

$$u(r) = \sum_{n=0}^{\infty} u_n(r), \ \ v(r) = \sum_{n=0}^{\infty} v_n(r), \tag{10}$$

and

$$f_1(u(\tau), v(\tau)) = \sum_{n=0}^{\infty} A_{1,n}(\tau), \ \ f_2(u(\tau), v(\tau)) = \sum_{n=0}^{\infty} A_{2,n}(\tau), \tag{11}$$

where the two-variable Adomian polynomials $A_{1,n}(\tau)$ are defined as

$$A_{1,n}(\tau) = \frac{1}{n!}\frac{d^n}{d\lambda^n} f_1\left(\sum_{n=0}^{\infty} u_n(\tau)\lambda^n, \sum_{n=0}^{\infty} v_n(\tau)\lambda^n\right)\Bigg|_{\lambda=0}, \tag{12}$$

and the two-variable Adomian polynomials $A_{2,n}(\tau)$ have similar expression. For new, convenient algorithms and MATHEMATICA subroutines of the multivariable Adomian polynomials see [5,7].

Upon substitution of the decomposition series into Eqs. (8) and (9), we obtain

$$\sum_{n=0}^{\infty} u_n(r) = \beta_1 + \int_0^1 \tau(1-\tau)\sum_{n=0}^{\infty} A_{1,n}(\tau)\, d\tau - \int_0^r \tau\left(1-\frac{\tau}{r}\right)\sum_{n=0}^{\infty} A_{1,n}(\tau)\, d\tau, \tag{13}$$

$$\sum_{n=0}^{\infty} v_n(r) = \beta_2 + \int_0^1 \tau(1-\tau)\sum_{n=0}^{\infty} A_{2,n}(\tau)\, d\tau - \int_0^r \tau\left(1-\frac{\tau}{r}\right)\sum_{n=0}^{\infty} A_{2,n}(\tau)\, d\tau. \tag{14}$$

Integrating term by term yields

$$\sum_{n=0}^{\infty} u_n(r) = \beta_1 + \sum_{n=0}^{\infty} \int_0^1 \tau(1-\tau) A_{1,n}(\tau)\, d\tau - \sum_{n=0}^{\infty} \int_0^r \tau\left(1-\frac{\tau}{r}\right) A_{1,n}(\tau)\, d\tau, \tag{15}$$

$$\sum_{n=0}^{\infty} v_n(r) = \beta_2 + \sum_{n=0}^{\infty} \int_0^1 \tau(1-\tau) A_{2,n}(\tau)\, d\tau - \sum_{n=0}^{\infty} \int_0^r \tau\left(1-\frac{\tau}{r}\right) A_{2,n}(\tau)\, d\tau. \tag{16}$$

We set the system of two-coupled Duan-Rach modified recursion schemes as [8]

$$u_0(r) = \beta_1,\ v_0(r) = \beta_2, \tag{17}$$

$$u_{n+1}(r) = \int_0^1 \tau(1-\tau) A_{1,n}(\tau)\, d\tau - \int_0^r \tau\left(1-\frac{\tau}{r}\right) A_{1,n}(\tau)\, d\tau,\ n \geq 0, \tag{18}$$

$$v_{n+1}(r) = \int_0^1 \tau(1-\tau) A_{2,n}(\tau)\, d\tau - \int_0^r \tau\left(1-\frac{\tau}{r}\right) A_{2,n}(\tau)\, d\tau,\ n \geq 0. \tag{19}$$

We set the system of two-coupled Duan's parametrized recursion schemes with decomposition of the two convergence parameters as [6,8,10]

$$c_1 = \sum_{n=0}^{\infty} c_{1,n},\ c_2 = \sum_{n=0}^{\infty} c_{2,n}, \tag{20}$$

$$u_0(r) = \beta_1 - c_1,\ v_0(r) = \beta_2 - c_2, \tag{21}$$

$$u_{n+1}(r) = c_{1,n} + \int_0^1 \tau(1-\tau) A_{1,n}(\tau)\, d\tau - \int_0^r \tau\left(1-\frac{\tau}{r}\right) A_{1,n}(\tau)\, d\tau,\ n \geq 0, \tag{22}$$

$$v_{n+1}(r) = c_{2,n} + \int_0^1 \tau(1-\tau) A_{2,n}(\tau)\, d\tau - \int_0^r \tau\left(1-\frac{\tau}{r}\right) A_{2,n}(\tau)\, d\tau,\ n \geq 0. \tag{23}$$

Then we obtain the two approximate solutions as

$$\phi_{m+1}(r) = \sum_{n=0}^{m} u_n(r),\ \psi_{m+1}(r) = \sum_{n=0}^{m} v_n(r). \tag{24}$$

If the exact solution is unknown, which is most often the case for nonlinear engineering equations, and boundary value problems in catalytic diffusion reactions, we compute the following error remainder functions and the maximal error remainder parameters as the error analysis for the sequence of approximate solutions. The error remainder functions and the maximal error remainder parameters are

$$ER_n^{\langle 1\rangle}(r) = \phi_n''(r) + \frac{2}{r}\phi_n'(r) + f_1(\phi_n(r), \psi_n(r)), \tag{25}$$

$$ER_n^{\langle 2\rangle}(r) = \psi_n''(r) + \frac{2}{r}\psi_n'(r) + f_2(\phi_n(r), \psi_n(r)), \tag{26}$$

and

$$MER_n^{\langle 1\rangle} = \max_{0\le r\le 1}\left|ER_n^{\langle 1\rangle}(r)\right|, \quad MER_n^{\langle 2\rangle} = \max_{0\le r\le 1}\left|ER_n^{\langle 2\rangle}(r)\right|, \tag{27}$$

which are a measure of how well the sequence of solution approximations satisfy the original nonlinear differential equation.

3 The boundary value problem in catalytic diffusion reactions

We solve the mixed BVP for the coupled Lane–Emden equations with the quadratic and product nonlinearities as [11]

$$u''(r) + \frac{2}{r}u'(r) - k_{11}u^2(r) - k_{12}u(r)\,v(r) = 0, \tag{28}$$

$$v''(r) + \frac{2}{r}v'(r) - k_{21}u^2(r) - k_{22}u(r)\,v(r) = 0, \tag{29}$$

$$u'(0) = 0,\ u(1) = \beta_1,\ v'(0) = 0,\ v(1) = \beta_2, \tag{30}$$

where we make the correspondences for the nonlinearities as

$$f_1(u(r), v(r)) = -k_{11}u^2(r) - k_{12}u(r)\,v(r), \tag{31}$$

$$f_2(u(r), v(r)) = -k_{21}u^2(r) - k_{22}u(r)\,v(r). \tag{32}$$

The boundary value problem (28)–(30) occurs in catalytic diffusion reactions [11]. The system parameters $\beta_1,\ \beta_2,\ k_{11},\ k_{12},\ k_{21}$ and k_{22} can be specified for the actual chemical reactions. Flockerzi and Sundmacher [11] considered the qualitative analysis for the solutions.

For the common quadratic nonlinearity, we have

$$u^2(r) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} u_{n-m}(r)\,u_m(r), \tag{33}$$

and for the common product nonlinearity, we have

$$u\left(r\right)v\left(r\right)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}u_{n-m}\left(r\right)v_{m}\left(r\right), \tag{34}$$

where the corresponding Adomian polynomials are

$$B_n=\sum_{m=0}^{n}u_{n-m}\left(r\right)u_m\left(r\right) \text{ and } C_n=\sum_{m=0}^{n}u_{n-m}\left(r\right)v_m(r), \tag{35}$$

respectively. Thus the two Adomian polynomials $A_{1,n}$ and $A_{2,n}$ are

$$A_{1,n}=-k_{11}B_n-k_{12}C_n,\ A_{2,n}=-k_{21}B_n-k_{22}C_n. \tag{36}$$

By the recursion schemes (17)–(19), we calculate that

$$\begin{aligned}
&u_0\left(r\right)=\beta_1,\, v_0\left(r\right)=\beta_2,\\
&u_1\left(r\right)=\frac{1}{6}\left(-1+r^2\right)\beta_1\left(k_{11}\beta_1+k_{12}\beta_2\right),\\
&v_1\left(r\right)=\frac{1}{6}\left(-1+r^2\right)\beta_1\left(k_{21}\beta_1+k_{22}\beta_2\right),\\
&u_2(r)=\frac{1}{360}\left(7-10r^2+3r^4\right)\beta_1\left(2k_{11}^2\beta_1^2+3k_{11}k_{12}\beta_1\beta_2+\right.\\
&\qquad\left.k_{12}\left(k_{21}\beta_1^2+\beta_2\left(k_{22}\beta_1+k_{12}\beta_2\right)\right)\right),\\
&v_2(r)=\frac{1}{360}\left(7-10r^2+3r^4\right)\beta_1\left(k_{22}\beta_2\left(k_{22}\beta_1+k_{12}\beta_2\right)+k_{21}\beta_1\left(k_{22}\beta_1+2k_{12}\beta_2\right)\right.\\
&\qquad\left.+k_{11}\beta_1\left(2k_{21}\beta_1+k_{22}\beta_2\right)\right),\ldots.
\end{aligned}$$

We take $\beta_1=1, \beta_2=2, k_{11}=1, k_{12}=2/5, k_{21}=1/2, k_{22}=1$, and calculate the sequences of the approximate solutions as

$$\begin{aligned}
&\phi_2(r)=\frac{7}{10}+\frac{3r^2}{10},\\
&\phi_3(r)=\frac{7357}{9000}+\frac{119r^2}{900}+\frac{151r^4}{3000},\\
&\phi_4(r)=\frac{1443251}{1890000}+\frac{59563r^2}{270000}+\frac{761r^4}{90000}+\frac{4609r^6}{630000},\\
&\phi_5(r)=\frac{128096033}{162000000}+\frac{9799289r^2}{56700000}+\frac{978857r^4}{27000000}-\frac{44143r^6}{56700000}+\frac{222571r^8}{226800000},\ldots,
\end{aligned}$$

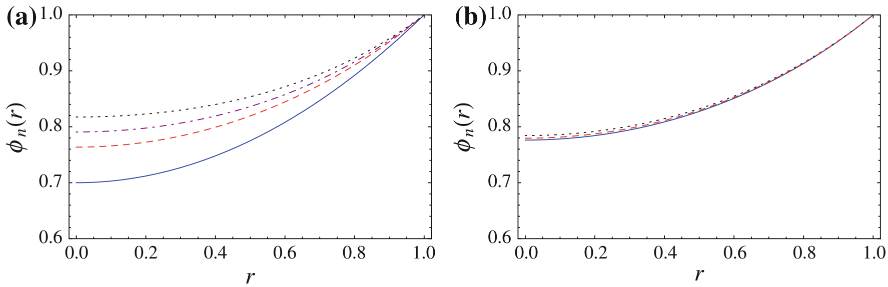

Fig. 1 The *curves* of $\phi_n(r)$ versus r for **a** $n = 2$ (*solid line*), $n = 3$ (*dot line*), $n = 4$ (*dash line*), $n = 5$ (*dot-dash line*); **b** $n = 6$ (*solid line*), $n = 7$ (*dot line*), $n = 8$ (*dash line*)

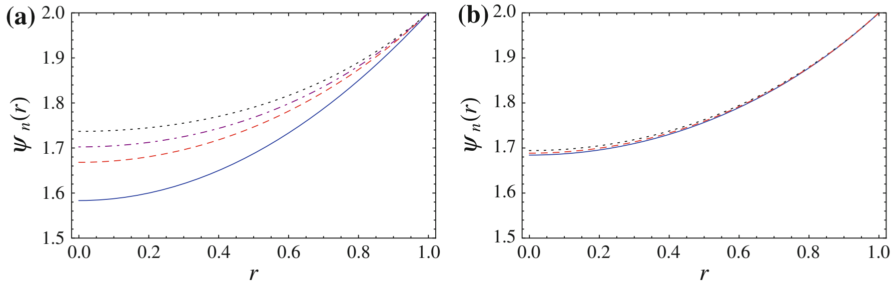

Fig. 2 The *curves* of $\psi_n(r)$ versus r for **a** $n = 2$ (*solid line*), $n = 3$ (*dot line*), $n = 4$ (*dash line*), $n = 5$ (*dot-dash line*); **b** $n = 6$ (*solid line*), $n = 7$ (*dot line*), $n = 8$ (*dash line*)

and

$$\psi_2(r) = \frac{19}{12} + \frac{5r^2}{12},$$

$$\psi_3(r) = \frac{6253}{3600} + \frac{71r^2}{360} + \frac{79r^4}{1200},$$

$$\psi_4(r) = \frac{1261171}{756000} + \frac{33467r^2}{108000} + \frac{457r^4}{36000} + \frac{2321r^6}{252000},$$

$$\psi_5(r) = \frac{772204247}{453600000} + \frac{5665609r^2}{22680000} + \frac{56881r^4}{1200000} - \frac{18271r^6}{22680000} + \frac{21919r^8}{18144000}, \ldots.$$

In Fig. 1, the curves of $\phi_n(r)$ versus r for $n = 2$ through 8 are plotted. In Fig. 2, the curves of $\psi_n(r)$ versus r for $n = 2$ through 8 are plotted.

Next, we calculate the error remainder functions

$$ER_n^{\langle 1\rangle}(r) = \phi_n''(r) + \frac{2}{r}\phi_n'(r) - k_{11}\phi_n^2(r) - k_{12}\phi_n(r)\psi_n(r), \quad (37)$$

$$ER_n^{\langle 2\rangle}(r) = \psi_n''(r) + \frac{2}{r}\psi_n'(r) - k_{21}\phi_n^2(r) - k_{22}\phi_n(r)\psi_n(r), \quad (38)$$

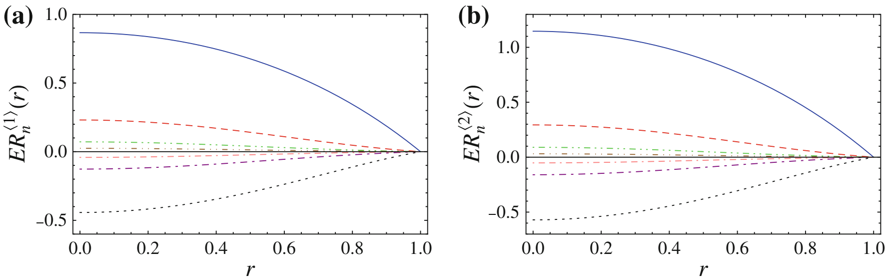

Fig. 3 The *curves* of the error remainder functions $ER_n^{\langle 1\rangle}(r)$ in **a** and $ER_n^{\langle 2\rangle}(r)$ in **b** for $n = 2$ (*solid line*), $n = 3$ (*dot line*), $n = 4$ (*dash line*), $n = 5$ (*dot-dash line*), $n = 6$ (*dot-dot-dash line*), $n = 7$ (*dot-dash-dash line*) and $n = 8$ (*dot-dot-dash-dash line*)

Table 1 The maximal error remainder parameters $MER_n^{\langle 1\rangle}$ and $MER_n^{\langle 2\rangle}$

n	$MER_n^{\langle 1\rangle}$	$MER_n^{\langle 2\rangle}$
2	0.866667	1.14667
3	0.442824	0.570630
4	0.230943	0.293825
5	0.126713	0.159885
6	0.0715579	0.0898227
7	0.0416048	0.0520260
8	0.0246787	0.0307762
9	0.0149069	0.0185503
10	0.00913022	0.0113429
11	0.00566118	0.00702355
12	0.00354550	0.00439385
13	0.00224019	0.00277360
14	0.00142608	0.00176425
15	0.000913892	0.00112983
16	0.000589065	0.000727815

and the maximal error remainder parameters

$$MER_n^{\langle 1\rangle} = \max_{0\le r\le 1}\left|ER_n^{\langle 1\rangle}(r)\right|, \quad MER_n^{\langle 2\rangle} = \max_{0\le r\le 1}\left|ER_n^{\langle 2\rangle}(r)\right|, \tag{39}$$

using MATHEMATICA. The curves of the error remainder functions $ER_n^{\langle 1\rangle}(r)$ and $ER_n^{\langle 2\rangle}(r)$ for $n = 2$ through 8 are plotted in Fig. 3.

The maximal error remainder parameters $MER_n^{\langle 1\rangle}$ and $MER_n^{\langle 2\rangle}$ are calculated by using the MATHEMATICA command 'NMaximize' and the values for $n = 2$ through 16 are listed in Table 1. The logarithmic plots of $MER_n^{\langle 1\rangle}$ and $MER_n^{\langle 2\rangle}$ for $n = 2$ through 16 are displayed in Fig. 4, where the dots are almost on a straight line, which demonstrates an approximate exponential rate of convergence. MATHEMATICA code generating Table 1 and Fig. 4 is attached in "Appendix".

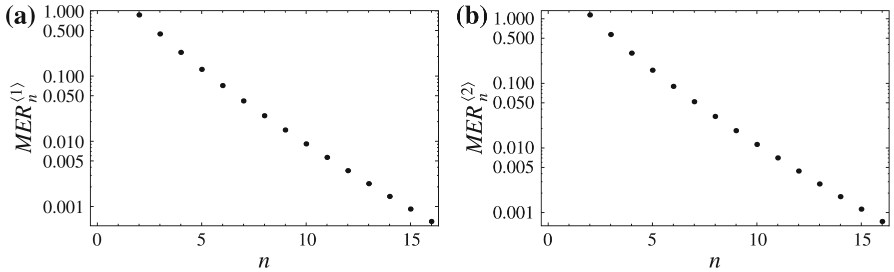

Fig. 4 Logarithmic plots of the maximal error remainder parameters $MER_n^{\langle 1 \rangle}$ in **a** and $MER_n^{\langle 2 \rangle}$ in **b** for $n = 2$ through 16

Next we consider the modified recursion scheme (20)–(23) with the parameters' choices $c_1 = c_2 = 0.15$, $c_{1,n} = c_{2,n} = 0.15/2^{n+1}$, $n = 0, 1, 2, \ldots$. The calculated solution approximations are listed below,

$$\phi_2(r) = \frac{2799}{4000} + \frac{901r^2}{4000},$$
$$\phi_3(r) = \frac{56117189}{72000000} + \frac{1080973r^2}{7200000} + \frac{791027r^4}{24000000},$$
$$\phi_4(r) = \frac{231724603307}{302400000000} + \frac{8389876921r^2}{43200000000} + \frac{238803197r^4}{14400000000} + \frac{420463703r^6}{100800000000}, \ldots,$$
$$\psi_2(r) = \frac{7693}{4800} + \frac{1547r^2}{4800},$$
$$\psi_3(r) = \frac{48923933}{28800000} + \frac{632581r^2}{2880000} + \frac{423419r^4}{9600000},$$
$$\psi_4(r) = \frac{202732690507}{120960000000} + \frac{4789506857r^2}{17280000000} + \frac{130778941r^4}{5760000000} + \frac{23940879r^6}{4480000000}, \ldots.$$

For the new solution approximations, the maximal error remainder parameters $MER_n^{\langle 1 \rangle}$ and $MER_n^{\langle 2 \rangle}$ are calculated and the values for $n = 2$ through 16 are listed in Table 2. The logarithmic plots of $MER_n^{\langle 1 \rangle}$ and $MER_n^{\langle 2 \rangle}$ for $n = 2$ through 16 are displayed in Fig. 5. We see that the new solution approximations have higher accuracy than the original solution approximations with the same term number.

4 Conclusions

In this work we proposed a reliable approach to handle a coupled Lane–Emden boundary value problems in catalytic diffusion reactions. We employed the Volterra integral forms, that we derived in [23,24] for the singular Lane–Emden ordinary differential equations. We then used the integral forms to formally derive the unprescribed initial condition $u(0)$ and $v(0)$. The powerful ADM was employed to determine the series solution that may converge to the exact solution if such a closed form solution exists, or we may use the obtained series for numerical computation. Numerical examples

Table 2 The new maximal error remainder parameters

n	$MER_n^{\langle 1\rangle}$	$MER_n^{\langle 2\rangle}$
2	0.413252	0.567430
3	0.236267	0.309872
4	0.0643409	0.0851094
5	0.0479165	0.0609052
6	0.0209370	0.0251111
7	0.0109410	0.0136470
8	0.00621286	0.00732628
9	0.00335204	0.00328331
10	0.00179895	0.00209486
11	0.000960968	0.00111331
12	0.000511238	0.000589608
13	0.000270998	0.000311281
14	0.000143189	0.000163880
15	0.0000754396	0.0000860597
16	0.0000396424	0.0000450898

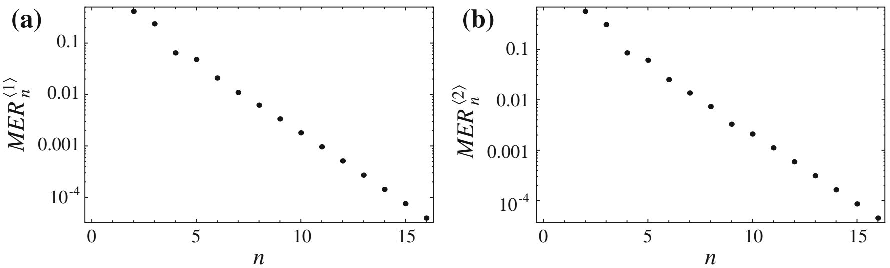

Fig. 5 Logarithmic plots of the new maximal error remainder parameters $MER_n^{\langle 1\rangle}$ in **a** and $MER_n^{\langle 2\rangle}$ in **b** for $n = 2$ through 16

are investigated to show the strength of the proposed scheme. The obtained results emphasize the efficiency of the new approach. We also demonstrate that a parametrized recursion scheme results in the solution approximations with higher accuracy.

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5 Appendix: MATHEMATICA code generating Table 1 and Fig. 4

```
A[n_]:=-k11*Sum[u[n-m]*u[m] ,{m,0,n}]-k12*Sum[u[n-m]*v[m] ,{m,0,n}];
B[n_]:=-k21*Sum[u[n-m]*u[m] ,{m,0,n}]-k22*Sum[u[n-m]*v[m] ,{m,0,n}];
u[0]=b1;  v[0]=b2;
b1=1; b2=2; k11=1;k12=2/5; k21=1/2;k22=1;
```

```
For[n=0,n<=14,n++,
  u[n+1]=Integrate[s (1-s) (A[n]/.r->s),{s,0,1}]-
  Integrate[s (1-s/r) (A[n]/.r->s),{s,0,r}];
  v[n+1]=Integrate[s (1-s) (B[n]/.r->s),{s,0,1}]-
  Integrate[s (1-s/r) (B[n]/.r->s),{s,0,r}];]
ph[n_]:=Sum[u[k],{k,0,n-1}];  ps[n_]:=Sum[v[k],{k,0,n-1}];
ER1[n_]:=D[ph[n],{r,2}]+2*Simplify[D[ph[n],r]/r]-
k11*ph[n]^2-k12*ph[n]*ps[n];
ER2[n_]:=D[ps[n],{r,2}]+2*Simplify[D[ps[n],r]/r]-
k21*ph[n]^2-k22*ph[n]*ps[n];
rr=Table[k1,{k1,2,16}];
da1=Table[NMaximize[{Abs[ER1[k1]],0<=r<=1},{r},AccuracyGoal->30,
PrecisionGoal->30,WorkingPrecision->30][[1]],{k1,2,16}];
da2=Table[NMaximize[{Abs[ER2[k1]],0<=r<=1},{r},AccuracyGoal->30,
PrecisionGoal->30,WorkingPrecision->30][[1]],{k1,2,16}];
F4a=ListLogPlot[Table[{rr[[n]],da1[[n]]},{n,1,Length[rr]}],
PlotRange->All,Frame->True];
F4b=ListLogPlot[Table[{rr[[n]],da2[[n]]},{n,1,Length[rr]}],
PlotRange->All,Frame->True];
{N[da1,6],N[da2,6],F4a,F4b}
```

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